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On the Construction of Test Problems for Concave Minimization Algorithms

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Abstract. We construct some classes of test problems of minimizing a concave or, more general, quasiconcave function over a polyhedral set. These test problems fulfil the general requirement that they have a global solution at a known point which is suitably chosen on the boundary of the feasible set.

Key words: Concave minimization, concave quadratic minimization, global optimization, test problem

The construction of nontrivial test problems in global optimization is an interesting problem, for which only a few results exist.

In Sung and Rosen (1982) and Kalantari and Rosen (1986) some classes of test problems for concave quadratic minimization problems are proposed. Pardalos (1987, 1991), Li and Pardalos (1992) and Hager et al. (1991) have generated test problems for the indefinite quadratic programs and related problems, (cf. also Pardalos and Rosen (1987) and Floudas and Pardalos (1990)). Each test problem mentioned above is constructed in a way so that it has a global solution at a given vertex of the feasible polytope. In this note we show the way to construct some classes of test problems of the form $\min\{f(x) : x \in D\}$, where D is a polyhedral set in $\mathbb{R}^n$ and f is a real-valued, concave or, more general, quasiconcave function defined on a suitable set of $\mathbb{R}^n$ which contains D. Each test problem generated here fulfils the general requirement that it has a global solution at a known point which is suitably chosen on the boundary of the feasible set.

In what follows we denote by D a nonempty polyhedral set in $\mathbb{R}^n$.

PROPOSITION 1. *Let $\ell(x)$ be a linear function on $\mathbb{R}^n$ such that the linear programs $\min\{\ell(x) : x \in D\}$ and $\max\{\ell(x) : x \in D\}$ have optimal solutions u^1 and u^2, respectively. Further, let φ be an arbitrary (quasi)concave function of one variable. Then the function $f(x) = \varphi(\ell(x))$ is (quasi)concave and the problem $\min\{f(x) = \varphi(\ell(x)) : x \in D\}$ has a global solution $u \in \{u^1, u^2\}$.*

Proof. First, we show that the function $f(x) = \varphi(\ell(x))$ is (quasi)concave on $\mathbb{R}^n$. Let x^1, x^2 be two points in $\mathbb{R}^n$ and $\lambda \in [0,1]$. Then we have $f(\lambda x^1 + (1-\lambda)x^2) = \varphi(\ell(\lambda x^1 + (1-\lambda)x^2)) = \varphi(\lambda\ell(x^1) + (1-\lambda)\ell(x^2)) \geq \lambda\varphi(\ell(x^1)) + (1-\lambda)\varphi(\ell(x^2)) = \lambda f(x^1) + (1-\lambda)f(x^2)$, if $\varphi(t)$ is concave, and $f(\lambda x^1 + (1-\lambda)x^2) = \varphi(\lambda\ell(x^1) + (1-\lambda)\ell(x^2)) \geq \min\{\varphi(\ell(x^1)), \varphi(\ell(x^2))\} = \min\{f(x^1), f(x^2)\}$, if

$\varphi(t)$ is quasiconcave. This implies that f is concave or quasiconcave if φ is concave or quasiconcave, respectively.

Now, again from the quasiconcavity of φ it follows that $\min\{f(x) : x \in D\} = \min\{\varphi(\ell(x)) : x \in D\} = \min\{\varphi(t) : \ell(u^1) \leq t \leq \ell(u^2)\} = \min\{\varphi(\ell(u^1)), \varphi(\ell(u^2))\} = \min\{f(u^1), f(u^2)\}$. This implies that Problem $\min\{f(x) = \varphi(\ell(x)) : x \in D\}$ has a global solution $u \in \{u^1, u^2\}$. □

EXAMPLE 1 (Concave quadratic test problem). Let D be defined by $D = \{x \in \mathbb{R}^n : a_i x \leq b\ (i = 1, \cdots, m)\}$, where $a_i \in \mathbb{R}^n$ $(i = 1, \cdots, m)$, $b \in \mathbb{R}^m$. Let v be a point on the boundary of D and $I(v) = \{i : a_i v = b_i\}$. Define $\ell(x) = ax$, where $a = \sum_{i \in I(v)} \lambda_i a_i, 0 \leq \lambda_i \leq 1\ (i \in I(v)), \sum_{i \in I(v)} \lambda_i = 1$, and define $\varphi(t) = -t^2$. Then, whenever the linear program $\min\{ax : x \in D\}$ has an optimal solution w, (e.g. when D is bounded), the concave quadratic minimization problem $\min\{f(x) = -(ax)^2 : x \in D\}$ has a global solution $u \in \{v, w\}$.

EXAMPLE 2. Let D be a polytope (bounded polyhedral set) in $\mathbb{R}^n$ and $\ell(x)$ an arbitrary linear function on $\mathbb{R}^n$. Then the linear programs $\min\{\ell(x) : x \in D\}$ and $\max\{\ell(x) : x \in D\}$ have optimal solutions u^1 and u^2, respectively. Defining $\varphi_1(t) = -|t|^{\frac{3}{2}}$ and $\varphi_2(t) = -|t|^{\frac{1}{2}}$, we obtain a concave function $f_1(x) = -|\ell(x)|^{\frac{3}{2}}$ and a quasiconcave function $f_2(x) = -|\ell(x)|^{\frac{1}{2}}$, and the problems

$$\min\{f_i(x) : x \in D\}\ (i = 1, 2)$$

have a global solution $u \in \{u^1, u^2\}$.

Other examples of this test problem type can be found in Horst and Thoai (1989), (cf. also Horst *et al.* (1991) and Horst and Tuy (1993)).

PROPOSITION 2. *Let $c(x)$ be a concave function and assume that the concave programming problem* $\min\{c(x) : x \in D\}$ *has a global solution u. Further, let $\varphi(t)$ be a concave nondecreasing function of one variable. Then the function $f(x) = \varphi(c(x))$ is concave, and problem* $\min\{f(x) = \varphi(c(x)) : x \in D\}$ *has a global solution u.*

Proof. We first show the concavity of $f(x)$. Since $c(x)$ is concave we have $c(\lambda x^1 + (1 - \lambda)x^2) \geq \lambda c(x^1) + (1 - \lambda)c(x^2)$ for $x^1, x^2 \in \mathbb{R}^n$, $0 \leq \lambda \leq 1$. Therefore, since $\varphi(t)$ is concave and nondecreasing it follows that $f(\lambda x^1 + (1 - \lambda)x^2) = \varphi(c(\lambda x^1 + (1 - \lambda)x^2) \geq \varphi(\lambda c(x^1) + (1 - \lambda)c(x^2)) \geq \lambda\varphi(c(x^1)) + (1 - \lambda)\varphi(c(x^2)) = \lambda f(x^1) + (1 - \lambda)f(x^2)$, i.e. f is concave.

Now, again since $\varphi(t)$ is nondecreasing we have

$$\begin{aligned} \min\{f(x) : x \in D\} &= \min\{\varphi(c(x)) : x \in D\} = \min\{\varphi(t) : c(u) \leq t\} \\ &= \varphi(c(u)) = f(u), \end{aligned}$$

which implies that u is a global solution of Problem $\min\{f(x) = \varphi(c(x)) : x \in D\}$. □

EXAMPLE 3. Let $g(x)$ be concave function, and let u be a known global solution of the concave program $\min\{g(x) : x \in D\}$ (e.g., generated as in examples 1, 2 above or as in Sung and Rosen (1982)). Further, let α be a real number satisfying $\alpha < g(u)$. Then, obviously the function $c(x) = g(x) - \alpha$ is concave and $F = \{x : c(x) \geq 0\}$ is a convex set containing D. Moreover, u is also a global solution of the program $\min\{c(x) : x \in D\}$.

Next, define a concave nondecreasing function $\varphi(t)$ by

$$\varphi(t) = \begin{cases} -\infty, & \text{if } t < 0 \\ \frac{t}{1+t} + log(1+t), & \text{if } t \geq 0 \end{cases}$$

Then the function

$$f(x) = \frac{c(x)}{1 + c(x)} + log(1 + c(x))$$

is concave on F, and a global solution of the concave program $\min\{f(x) : x \in D\}$ occurs at u.

PROPOSITION 3. *If the function* $c(x)$ *is quasiconcave on a convex set* $A \subset \mathbb{R}^n$ *and* $\varphi(t)$ *is an arbitrary nondecreasing function of one variable, then the function* $f(x) = \varphi(c(x))$ *is quasiconcave on* A*, and every global solution of problem* $\min\{c(x) : x \in D \subset A\}$ *is also a global solution of problem* $\min\{f(x) : x \in D\}$.

Proof. From the quasiconcavity of $c(x)$ it follows that for $x^1, x^2 \in A$ and $0 \leq \lambda \leq 1$ we have $c(\lambda x^1 + (1-\lambda)x^2) \geq \min\{c(x^1), c(x^2)\}$. Therefore, since φ is nondecreasing we have $f(\lambda x^1 + (1-\lambda)x^2) = \varphi(c(\lambda x^1 + (1-\lambda)x^2)) \geq \varphi(min\{c(x^1), c(x^2)\}) = \min\{\varphi(c(x^1)), \varphi(c(x^2))\} = \min\{f(x^1), f(x^2)\}$, which implies that f is quasiconcave on A.

Now, let $u \in D \subset A$ such that $c(u) = \min\{c(x) : x \in D\}$. Then, since φ is nondecreasing, it follows that $\min\{f(x) : x \in D\} = \min\{\varphi(c(x)) : x \in D\} = \min\{\varphi(t) : c(u) \leq t\} = \varphi(c(u)) = f(u)$, i.e., u is a global solution of problem $\min\{f(x) : x \in D\}$. □

EXAMPLE 4. Let $c(x) = \prod_{i=1}^{n} x_i$. It can be shown that $c(x)$ is quasiconcave on the set $A = \{x \in \mathbb{R}^n : x > 0\}$. Define a nondecreasing function

$$\varphi(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ t^2 + t^{\frac{1}{2}}, & \text{if } t > 0 \end{cases}$$

Then the function $f(x) = \prod_{i=1}^{n} x_i^2 + \prod_{i=1}^{n} x_i^{\frac{1}{2}}$ is quasiconcave in A, and each global solution of problem $\min\{\prod_{i=1}^{n} x_i : x \in D\}$ with $D \subset A$ is also a global solution of problem $\min\{\prod_{i=1}^{n} x_i^2 + \prod_{i=1}^{n} x_i^{\frac{1}{2}} : x \in D\}$.

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